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ASYMPTOTICS OF THE EIGENVALUES AND EIGENFUNCTIONS IN CUBICALLY ANISOTROPIC THERMOELASTIC BODIES WITH SLITS

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Within the framework of the uncoupled thermoelasticity, using G. Weyl's method, asymptotic formulas for eigenvalues and eigenfunctions of the first boundary-value problem have been obtained for cubically anisotropic bodies limited by a finite number of closed and nonclosed unintersecting Lyapunov surfaces.

Introduction. Asymptotic spectral laws play an important role in problems of radiation, theory of elastic vibrations, specific heat of solids, thermodynamics, and in other branches of mathematical physics. At the present time their role has substantially increased thanks to the introduction of new structural materials into design practice. In the present work these problems are considered for bodies with cubic anisotropy, the elastic properties of which are characterized by three material constants.

Statement of the Problem. Let D be a region in the three-dimensional Euclidean space E_3 whose boundary S consists of a finite number of closed and nonclosed Lyapunov surfaces $S = \Sigma \cup \sigma$, where Σ is a finite number of closed surfaces, $\Sigma \underset{k}{\cup} \Sigma_k$, $k - \overline{0, K}$; σ is a finite number of nonclosed surfaces $\sigma \underset{n}{\cup} \sigma_n$, $n = \overline{1, N}$, $k - \overline{1, K}$. We will assume that σ_n are bounded by smooth curves Γ_n. Moreover, we assume that the region is filled with a cubically anisotropic medium the elastic properties of which are characterized by the following Duhamel–Neumann law [1]:

$$\sigma_{ii} = (A_{11} - A_{12})\,\varepsilon_{ii} + A_{12}\varepsilon - \beta\theta\,, \quad \sigma_{ij} = 2A_{44}\varepsilon_{ij} \quad i \neq j = 1, 2, 3\,. \tag{1}$$

The tensor deformation components ε_{ij} are related to the components of the displacement vector u_i of the elastic body points by the formulas

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)\,, \quad \partial_i = \frac{\partial}{\partial x_i}\,, \quad \varepsilon = \sum_{i=1}^{3} \partial_i u_i\,, \quad j, i = 1, 2, 3\,. \tag{2}$$

The equations of thermoelastic equilibrium of the cubically anisotropic body have the form

$$\sum_{j=1}^{3} \partial_j \sigma_{ij} + F'_\alpha = 0\,, \quad i = 1, 2, 3\,. \tag{3}$$

Here, $F'_\alpha = F_\alpha - \partial_\alpha(\beta\theta)$ and F_α denotes the components of the external force related to a unit volume. Then, by virtue of formulas (1), (2), from Eq. (3) we obtain

$$\left[A_{44}\Delta + (A_{11} - A_{12} - 2A_{44})\,\partial_\alpha^2\right] u_\alpha + (A_{12} + A_{44})\,\partial_\alpha \varepsilon + F'_\alpha = 0 \tag{4}$$

or in matrix form

$$M \cdot u + F' = 0\,, \tag{5}$$

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where

$$M \cdot u = A_{44} \begin{Vmatrix} \Delta + (\varepsilon + \sigma)\, \partial_\alpha^2 & \sigma \partial_\beta \partial_\alpha & \sigma \partial_\gamma \partial_\alpha \\ \sigma \partial_\beta \partial_\alpha & \Delta + (\varepsilon + \sigma)\, \partial_\beta^2 & \sigma \partial_\beta \partial_\alpha \\ \sigma \partial_\gamma \partial_\alpha & \sigma \partial_\beta \partial_\gamma & \Delta + (\varepsilon + \sigma)\, \partial_\gamma^2 \end{Vmatrix} \cdot \begin{Vmatrix} u_\alpha \\ u_\beta \\ u_\gamma \end{Vmatrix} \; . \tag{6}$$

Here, $\varepsilon = \dfrac{A_{11} - A_{12}}{A_{44}} - 2$, $\sigma = 1 + \dfrac{A_{12}}{A_{44}}$. Since ε and σ are constants, the equation

$$\left[M \left(\frac{\partial}{\partial x} \right) - \lambda^2 E \right] u\,(x) = - F' \;, \quad \lambda = \text{const} \tag{7}$$

can be solved with the aid of Fourier transformation:

$$u\,(x) = \iiint\limits_{\infty} \tilde{u}\,(\alpha) \exp\,(i \alpha x)\, d\alpha \;, \quad F'(x) = \iiint\limits_{\infty} \tilde{F}\,(\alpha) \exp\,(i \alpha x)\, d\alpha \;; \tag{8}$$

$$\tilde{u}\,(\alpha) = \frac{1}{(2\pi)^3} \iiint\limits_{\infty} u\,(x) \exp\,(-\, i\,(\alpha, x))\, dx \;, \quad \tilde{F}\,(\alpha) = \frac{1}{(2\pi)^3} \iiint\limits_{\infty} F'(x) \exp\,(-\, i\,(\alpha, x))\, dx \;. \tag{9}$$

Substituting (8) into (7) and taking into account that M is the second-order operator matrix, we obtain

$$M\,(i\alpha) = -\, M\,(\alpha) \;, \quad \iiint\limits_{\infty} \exp\,(i\,(\alpha, x)) \left[(M\,(\alpha) + \lambda^2 E)\, \tilde{u}\,(\alpha) - \tilde{F}\,(\alpha) \right] d\alpha = 0 \;,$$

whence

$$(M\,(\alpha) + \lambda^2 E)\, \tilde{u}\,(\alpha) - \tilde{F}\,(\alpha) = 0$$

and therefore

$$\tilde{u}\,(\alpha) = (M\,(\alpha) + \lambda^2 E)^{-1}\, \tilde{F}\,(\alpha) \;, \tag{10}$$

where the (−1) power denotes the inverse matrix, each element of which represents a cofactor of matrix (6) divided by the determinant det $(M(\alpha) + \lambda^2 E) = 0$:

$$\Phi\,(\partial_1, \partial_2, \partial_3) = \begin{Vmatrix} (\Delta^*)^2 + \lambda \Delta^* (\partial_2^2 + \partial_3^2) + (\lambda^2 - \sigma^2)\, \partial_2^2 \partial_3^2 & -\, \sigma \partial_1 \partial_2\, [\Delta^* + (\lambda - \sigma)\, \partial_3^2] & -\, \sigma \partial_1 \partial_3\, [\Delta^* + (\lambda - \sigma)\, \partial_2^2] \\ -\, \sigma \partial_1 \partial_2\, [\Delta^* + (\lambda - \sigma)\, \partial_3^2] & (\Delta^*)^2 + \lambda \Delta^* (\partial_1^2 + \partial_3^2) + (\lambda^2 - \sigma^2)\, \partial_1^2 \partial_3^2 & -\, \sigma \partial_2 \partial_3\, [\Delta^* + (\lambda - \sigma)\, \partial_1^2] \\ -\, \sigma \partial_1 \partial_3\, [\Delta^* + (\lambda - \sigma)\, \partial_2^2] & -\, \sigma \partial_2 \partial_3\, [\Delta^* + (\lambda - \sigma)\, \partial_1^2] & (\Delta^*)^2 + \lambda \Delta^* (\partial_1^2 + \partial_2^2) + (\lambda^2 - \sigma^2)\, \partial_1^2 \partial_1^2 \end{Vmatrix},$$

$$F\,(\partial_1, \partial_2, \partial_3) = \det \left(M\,(\alpha)\, \lambda^2 E \right) = \Delta^3 + \lambda f_1 \Delta^2 + \left(\lambda^2 - \sigma^2 \right) f_2 \Delta + (\lambda - \sigma)^2\, (\lambda + 2\sigma)\, f_3 \;,$$

$$f_1 = \partial_1^2 + \partial_2^2 + \partial_3^2 = \Delta \;, \quad f_2 = \partial_1^2 \partial_2^2 + \partial_2^2 \partial_3^2 + \partial_3^2 \partial_1^2 \;, \quad f_3 = \partial_1^2 \partial_2^2 \partial_3^2 \;.$$

It is assumed that this inverse matrix exists, i.e., λ represents the roots of the equation det $(M(\alpha) + \lambda^2 E) = 0$ and they are complex numbers. Therefore

$$u\,(x) = \iiint\limits_{\infty} \exp\,(i\,(\alpha, x)) \left(M\,(\alpha) + \lambda^2 E \right)^{-1} \tilde{F}\,(\alpha)\, d\alpha = 0 \;. \tag{11}$$

We substitute Eq. (9) into Eq. (11) and introduce the notation

$$g(x, \lambda) = \frac{1}{(2\pi)^3} \iiint_\infty \exp\left(i\,(\alpha, x)\right) \left(M(\alpha) + \lambda^2 E\right)^{-1} d\alpha \,. \tag{12}$$

Then Eq. (11) takes the form

$$u(x) = \iiint_\infty g(x - y, \alpha)\, F(y)\, dy \,. \tag{13}$$

The quantity $u(x)$ defined in this way is the improper integral of the first and second kind. It follows from Eq. (13) that $g(x - y, \lambda)$ represents a fundamental matrix for the system of equations (5) from the viewpoint of definition [2]. From its structure it follows that it is continuous and twice differentiable with respect to x at $x \neq y$ and satisfies system (5) with respect to x, and the particular solution of system (5) can be represented in the form of (13). Actually we have

$$g(x, \lambda) = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \frac{1}{i\,(2\pi)^3\,|x|^2} \iiint_{|\alpha| < R_1} \left[\left(x_1 \frac{\partial}{\partial \alpha_1} + x_2 \frac{\partial}{\partial \alpha_2} + x_3 \frac{\partial}{\partial \alpha_3}\right)\right] \exp\left(i\lambda\,(\alpha, x)\right) \left(M(\alpha) + E\right)^{-1} d\alpha$$

$$= \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \frac{1}{(2\pi)^3\,|x|^2} \iiint_{|\alpha| < R_1} \exp\left(i\lambda\,(\alpha, x)\right) \left[\left(x_1 \frac{\partial}{\partial \alpha_1} + x_2 \frac{\partial}{\partial \alpha_2} + x_3 \frac{\partial}{\partial \alpha_3}\right) \left(M(\alpha) + E\right)^{-1}\right] d\alpha \,,$$

where $R_1 = R/\lambda$; $\alpha = \rho\xi$.

Each element of the matrix $M(\alpha)$ represents the ratio of two polynomials in ρ of the fourth degree (numerator) and sixth degree (denominator). Therefore

$$\left(M(\alpha) + E\right)^{-1} = O\left(\frac{1}{1 + \rho^2}\right), \quad \left(x_1 \frac{\partial}{\partial \alpha_1} + x_2 \frac{\partial}{\partial \alpha_2} + x_3 \frac{\partial}{\partial \alpha_3}\right) \left(M(\alpha) + E\right)^{-1} = O\left(\frac{1}{1 + \rho^2}\right).$$

These formulas suggest the possibility of the limiting transition for $R \to \infty$ and $\varepsilon \to 0$, the performance of which yields

$$g(x, \lambda) = \frac{i}{(2\pi)^3\,|x|^2} \iiint_\infty \exp\left(i\lambda\,(\alpha, x)\right) \left[\left(x_1 \frac{\partial}{\partial \alpha_1} + x_2 \frac{\partial}{\partial \alpha_2} + x_3 \frac{\partial}{\partial \alpha_3}\right) \left(M(\alpha) + E\right)^{-1}\right] d\alpha \,.$$

Going over from the variables α_1, α_2, and α_3 to the variables $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}_3$ so that the positive direction of $\tilde{\alpha}_3$ could coincide with the vector x, we find

$$(\alpha, x) = |x|\,\tilde{\alpha}_3 \,, \quad x_1 \frac{\partial}{\partial \alpha_1} + x_2 \frac{\partial}{\partial \alpha_2} + x_3 \frac{\partial}{\partial \alpha_3} = |x|\,\frac{\partial}{\partial \tilde{\alpha}_3} \,, \quad \partial\alpha = \partial\tilde{\alpha} \,.$$

Therefore replacement of $\tilde{\alpha}$ by α yields

$$g(x, \lambda) = \frac{i}{(2\pi)^3\,|x|^2} \iint_{S_\infty} d\alpha_1 d\alpha_2 \int_{-\infty}^{+\infty} \exp\left(i\lambda\,|x|\,\alpha_3\right) \frac{\partial}{\partial \alpha_3} \left(M(\alpha) + E\right)^{-1} d\alpha_3 \,.$$

The inside integral contains an analytical function of α_3. In the region D we will consider the Dirichlet problem:

$$\left[M\left(\frac{\partial}{\partial x}\right)-\lambda^2 E\right]u(x)=-F', \quad x — D, \quad u(x)=0, \quad x — S, \quad \lambda=\text{const}. \tag{14}$$

The solution of problem (14) can be presented with the aid of Green's matrix. The existence of such a matrix in the case where the boundary of the region D consists of a finite number of Lyapunov-type closed nonintersecting surfaces $\Sigma \cup_k \Sigma_k$ follows from [2]. In the case where the boundary of the region D consists of a finite number of Lyapunov-type nonclosed nonintersecting surfaces its existence follows from the studies made in [3] that rest on the theory of two-values potentials which was constructed for the elliptic systems of variational type second-order equations.

The Green matrix for problem (14) will be sought in the form

$$G(x-y,\lambda)=g(x-y,\lambda)+g_1(x,y,\lambda), \tag{15}$$

where

$$g_1(x,y,\lambda)=\iiint_D g(x-z,\lambda)\,h(z,y,\lambda)\,dz; \tag{16}$$

$g(x-y, \lambda)$ is the fundamental matrix of system (14). The determination of $h(z, y, \lambda)$ is reduced to the solution of the system of integral equations

$$h(x,y,\lambda)=P(x,y,\lambda)+\iiint_D P(x,z,\lambda)\,h(z,y,\lambda)\,dz. \tag{17}$$

Here, $P(x, y, \lambda)$ is the result of the application of the operator $M\left(\frac{\partial}{\partial x}\right)-\lambda^2 E$ to Eq. (15), with the following estimate being valid in the vicinity of the point $x = y$ and at infinity [2]:

$$P(x,y,\lambda)=O\left(\frac{\exp(-\lambda\varepsilon\,|x-y|)}{|x-y|^2}\right). \tag{18}$$

It ensures the convergence of the improper integral in (17) at infinity. It follows from Eq. (17) that the system of equations (16) at high values of λ is unambiguously resolvable by the method of successive approximations. From the second Green's formula for system (14) the symmetry of Green's matrices $G(x, y) = G'(x, y)$ follows. Therefore integral equations (17) lead to the following equalities:

$$G_1(x,y,\lambda)=G(x,y)-\lambda^2\sum_{n=1}^{\infty}\frac{u_n(x)\,u_n'(x)}{\lambda_k(\lambda_n+\lambda^2)},$$

where λ_k are eigenvalues; $\{u_n(x)\}$ is the complete orthonormalized system of eigenfunctions of the argument $G(x, y)$. Therefore the following equality is valid:

$$\psi(x,\lambda)=\iiint_D G(x,y)\,G(y,x,\lambda)\,dy=\sum_{n=1}^{\infty}\frac{u_n(x)\,u_n'(x)}{\lambda_k(\lambda_n+\lambda^2)}. \tag{19}$$

From this, reasoning ordinarily, we arrive at the following equality [4, 5]:

$$\lim_{\lambda\to\infty}\sum_{n=1}^{\infty}\frac{u_n(x)\,u_n'(x)}{\lambda_k(\lambda_n+\lambda^2)}=\frac{1}{(2\pi)^3}\iiint_{D_\infty}\left[(M(x,\alpha))\,(M(x,\alpha)+E)\right]^{-1}d\alpha. \tag{20}$$

Since $u_n'(x)u_n(x)$ = Sp $u_n(x)u_n'(x)$, Eq. (20) can be transformed to give

$$\lim_{\lambda\to\infty}\sqrt{\lambda}\sum_{n=1}^{\infty}\frac{u_n(x)\,u_n'(x)}{\lambda_k(\lambda_n+\lambda^2)}=\frac{C^*(x)}{4\pi}, \tag{21}$$

where

$$C^*(x)=\frac{1}{2\pi^2}\iiint_{D_\infty}\mathrm{Sp}\left[(M(x,\alpha))\,(M(x,\alpha)+E)\right]^{-1}d\alpha\,.$$

Now, we shall avail ourselves of the following Tauber theorem [4]:

Theorem. *If the series* $s(\lambda)\sum_{k=1}^{\infty}\frac{c_k}{\lambda_k+\lambda}$, *where* $c_k\ge 0,\ 0\le\lambda_1\le\lambda_2\le\lambda_3\le\ldots\le\lambda_n\to\infty$, *converges* at $\lambda>0$ *and* $\lim_{\lambda\to\infty}\sqrt{\lambda}\,s(\lambda)=H$, *then* $\lim_{\lambda\to\infty}\frac{1}{\sqrt{\lambda}}\sum_{\lambda_k\le\lambda}c_k=\frac{2H}{\pi}$, *with summation in the last sum being made by those values of k for which* $\lambda_k\le\lambda$.

In the case considered by us $c_k=\frac{a_k(x)}{\lambda_k}$, $H=\frac{C^*(x)}{4\pi}$. Therefore

$$\lim_{\lambda\to\infty}\frac{1}{\sqrt{\lambda}}\sum_{\lambda_k\le\lambda}\frac{a_k(x)}{4\pi}=\frac{C^*(x)}{2\pi^2} \tag{22}$$

or

$$\Phi(x,\lambda)\equiv\sum_{\lambda_k\le\lambda}\frac{a_k(x)}{\lambda_k}=\frac{C^*(x)}{2\pi^2}\sqrt{\lambda}+\varepsilon(\lambda)\sqrt{\lambda}\,. \tag{23}$$

Here $\varepsilon(\lambda)\to 0$, when $\lambda\to\infty$. Assuming in (22) that $\lambda=\lambda_n$, we obtain

$$\sigma_n(x)\equiv\sum_{k=1}^{n}\frac{a_k(x)}{\lambda_k}=\frac{C^*(x)}{2\pi^2}\sqrt{\lambda_n}+\varepsilon_n\sqrt{\lambda_n}\,, \tag{24}$$

where $\varepsilon_n\to 0$ for $n\to\infty$; $\Phi(x,\lambda)$ is the nondecreasing function of λ with $\Phi(x,\lambda)=0$ at $\lambda<\lambda_1$ and $\Phi(x,\lambda)=\sigma_m(x)$ at $\lambda_m\le\lambda<\lambda_{m+1}$. The integrability of $\Phi(x,\lambda)$ over λ at any finite interval of change in λ follows from Eq. (23) with

$$\int_0^{\lambda_n}\Phi(x,\lambda)\,d\lambda=\frac{C^*(x)}{3\pi^2}\lambda_n^{3/2}+\int_0^{\lambda_n}\varepsilon\sqrt{\lambda}\,d\lambda\,.$$

Consequently

$$\sum_{k=1}^{n}u_k'(x)\,u_k(x)\sim\frac{1}{12\pi^4}\iiint_D\mathrm{Sp}\left[(M(\alpha))\,(M(\alpha)+E)\right]^{-1}d\alpha\lambda_n^{3/2},$$

$$n \sim \frac{v}{12\pi^4} \iiint_D \mathrm{Sp}\left[(M(\alpha))\,(M(\alpha)+E)\right]^{-1} d\alpha \lambda_n^{3/2}.$$

For actual calculations it is worthwhile to go over in these formulas to spherical coordinates with the center at their beginning. As a result, we obtain the asymptotic equalities

$$\sum_{k=1}^{n} u_k'(x)\, u_k(x) \sim \frac{i}{12\pi^3} \iiint_D \sum \mathrm{res}\,\mathrm{Sp}\left[(M(\alpha))\,(M(\alpha)\,\rho^3+E)\right]^{-1} d\alpha \lambda_n^{3/2},$$

$$n \sim \frac{iv}{12\pi^3} \iiint_D \sum \mathrm{res}\,\mathrm{Sp}\left[(M(\alpha))\,(M(\alpha)\,\rho^3+E)\right]^{-1} d\alpha \lambda_n^{3/2}.$$

According to [5], from these formulas it follows that

$$\sum_{k=1}^{n} \left[u_k(x)\right]^2 \sim \frac{1}{(2\pi)^3} \iiint_{a(x,\alpha)<1} d\alpha \lambda_n^{3/2}, \quad n \sim \frac{1}{(2\pi)^3} \iiint_D \left\{ \iiint_{a(x,\alpha)<1} d\alpha \right\} dx \lambda_n^{3/2}.$$

Conclusions. These asymptotic equalities were earlier obtained [4–6] for the regions bounded by closed smooth surfaces. For regions with nonclosed boundaries they have been given for the first time.

NOTATION

A_{ij}, material constants considered to be constant values; ds, element of surface S; M, operator matrix; $P(x)$, point E_3; $P(x, y, \lambda)$, matrix; S, surface in E_3; Sp, matrix trace; $u(x)$, column vector; v, volume of region D; β, material constants; θ, change in temperature; λ_n, eigenvalues of the Dirichlet problem.

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